MTH 264 Introduction to Matrix Algebra - Summer 2023.

LN1A. Vectors in $\mathbb{R}^n$, Vector Operations, and Geometric Interpretation.

These lecture notes are mostly lifted from the text **Matrix and Power Series, Lee and Scarborough, custom 5th edition**. This document highlights which parts of the text are used in the lecture sessions.

Part 1. Vector Spaces.

Definition 1A.1. Vector Spaces.

A **vector space** is a set composed of the following:

- **(a)** a collection of elements called **vectors**;
- **(b)** a collection of elements called **scalars**;
- **(c)** a **scalar multiplication** function that takes in a vector and a scalar and returns a vector;
- **(d)** a **vector addition** function that takes in two vectors and returns a vector;

such that certain properties called **vector space properties/axioms** are satisfied.

Note that this definition of a vector space is very abstract and if you take a linear algebra course, this is the definition you'll be working with. However, in this class, we'll focus on a specific vector space (i.e. $\mathbb{R}^n$ defined below) so we don't have to use this abstract definition.

Definition 1A.2. Vector Space Axioms/Properties

- **A1. Commutativity of Vector Addition.** For all vectors $\mathbf{u}, \mathbf{v}$: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
- **A2. Associativity of Vector Addition.** For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
- **A3. Zero Vector Identity.** There exists a **zero vector 0** such that for all vectors $\mathbf{u}$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- **A4. Additive Inverses.** Every vector $\mathbf{u}$ has an additive inverse denoted $-\mathbf{u}$ satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$;
- **A5. Scalar Multiplication Identity.** There exists a scalar 1 such that for all vectors $\mathbf{u}$, $1\mathbf{u} = \mathbf{u}$;
- **A6. Vector Distributivity over Scalar Addition.** For all vectors $\mathbf{u}$ and scalars a, b: $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
- **A7. Scalar Distributivity over Vector Addition.** For all vectors $\mathbf{u}, \mathbf{v}$ and scalars a, $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$;
- **A8. Scalar Associativity over Scalar Multiplication.** For all vectors $\mathbf{u}$ and scalars a, b, $(ab)\mathbf{u} = a(b\mathbf{u})$;

While it is not critical that you know the names of these properties, I've included them so it's easier to reference them in explanations. Additionally: if we were to talk about vector space axioms in the general sense, there is a lot of nuance in the choice of notation. For example, 1 refers to the identify of the set of scalars – which if the set of scalars is not $\mathbb{R}$ or $\mathbb{C}$, 1 may be different. There is also technically a difference between the additive inverse $-\mathbf{u}$ and the product $(-1)\mathbf{u}$ but it can be proven that those two are equal.

We won't test you on these nuances since this is not a linear algebra course. As said earlier, we'll be focusing on the vector space $\mathbb{R}^n$ so most of these properties should be relatively easy to see as a consequence of the properties of $\mathbb{R}$.

Definition 1A.3. Vectors in $\mathbb{R}^n$.

Fix $n \in \mathbb{N}$. The vector space $\mathbb{R}^n$ is the vector space with the set of real numbers $\mathbb{R}$ as **scalars** and the set of ordered lists of n real numbers as **vectors**. Note that vectors in $\mathbb{R}^n$ are typically denoted as $\mathbf{v} = (v_1, v_2, ..., v_n)$, i.e. components are indexed by the numbers $\{1, 2, ..., n\}$.

Vector addition and **scalar multiplication** on $\mathbb{R}^n$ are defined respectively below.

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n) \qquad \text{and} \qquad k\mathbf{u} = (kv_1, kv_2, ..., kv_n)$$

For brevity, we denote $\mathbf{v}$ being a vector as $v \in \mathbb{R}^n$ and k being a scalar as $k \in \mathbb{R}$.

For this course, we'll mainly focus on the case of $n = 2$ and $n = 3$ for graphical applications; and $n \leq 4$ for linear systems. Also, observe that we haven't defined vectors as either row vectors or column vectors (in the matrix sense). This is intentional since all the vector space properties are applicable regardless of presentation. When we consider matrix multiplication later, we'll justify why we prefer vectors to be column vector form.

Theorem 1A.4. $\mathbb{R}^n$ *is* a vector space.

The vector space $\mathbb{R}^n$, as defined above, satisfies all the vector space axioms with

(a) Zero vector is given by $\mathbf{0} = (0, 0, ..., 0)$, i.e. all components are zero;

(b) For a vector $\mathbf{u} = (u_1, u_2, ..., u_n)$, the additive inverse $-\mathbf{u}$ is given by $-\mathbf{u} = (-u_1, -u_2, ..., -u_n)$.

(c) The scalar identity is the usual $1 \in \mathbb{R}$.

Often, we also represent vectors as **linear combinations** of other vectors.

Definition 1A.5. Linear Combination and Span

Let $V = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ be vectors. Then, a **linear combination** of V is any vector of the form

$$a_1\mathbf{v_1} + a_2\mathbf{v_2} + ...a_n\mathbf{v_k} \qquad \text{for any scalars} \qquad a_1, a_2, ..., a_k.$$

The set of all linear combinations of V is called the **span** of V and is denoted by $\text{span}(V)$.

For $\mathbb{R}^2$ and $\mathbb{R}^3$, there is a special set of vectors that will allow us to use a different representation of vectors.

Definition 1A.6. Standard Basis Vectors

The **standard basis vectors** for $\mathbb{R}^2$ is the set $\{\mathbf{i}, \mathbf{j}\}$ with $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. Similarly, the **standard basis vectors** for $\mathbb{R}^3$ is the set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$.

Observe that there is some abuse of notation here since $\mathbf{i}$ can either represent $(1, 0)$ or $(1, 0, 0)$. However, if we consider that $\mathbf{i}$ is the vector in $\mathbb{R}^n$ with 1 as its first coordinate and 0 everywhere else, then our definitions of $\mathbf{i}$ agree for $\mathbb{R}^2$ and $\mathbb{R}^3$. In this course, when we refer to $\mathbf{i}$, it's usually clear from context if we're working in $\mathbb{R}^2$ or in $\mathbb{R}^3$. The same applies to $\mathbf{j}$.

Theorem 1A.7. Alternate Representation of Vectors in $\mathbb{R}^n$

Let $v = (a, b) \in \mathbb{R}^2$. Then, $v = a\mathbf{i} + b\mathbf{j}$. Similarly, let $w = (a, b, c) \in \mathbb{R}^3$. Then, $w = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

You may see in the homework exercises that we may switch between the coordinate-wise representation of vectors and the linear combination representation of vectors with respect to the standard basis vectors. We provide an example below.

Example 1A.7.1. The vector $(3t, 4t^2+1, 6t-3)$ for some $t \in \mathbb{R}$ may be expressed as $(3t)\mathbf{i}+(4t^2+1)\mathbf{j}+(6t-3)\mathbf{k}$.

While I personally do not prefer this notation, it's important that you know how to work with it when information is presented in this manner.

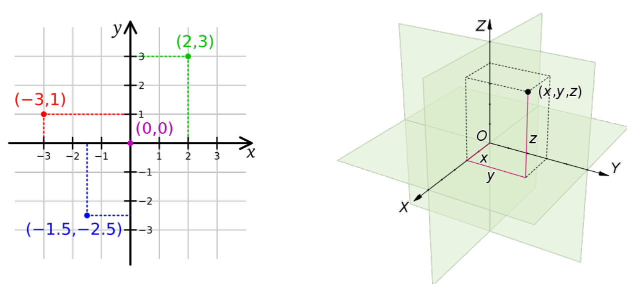
Part 2. Geometric Interpretation of Vectors.

Convention 1A.8. Graphical Interpretation of Vectors in $\mathbb{R}^n$.

We typically interpret vectors in $\mathbb{R}^n$ as coordinates in n-space (e.g. the Cartesian plane in the $n = 2$ case). When convenient, a vector (i.e. a set of coordinates) either represents a point or an arrow from the origin to said point. In the latter case, a vector carries direction information with direction determined by its components.

Since n-space admits vectors as coordinates, **we can talk about n-space and the vector space $\mathbb{R}^n$ interchangeably**.

In the following image, vectors are considered points with vector components as the coordinates. The image on the left is on the Cartesian plane, i.e. 2-space and the image on the left is on 3-space.



This gives us the following interpretations on vectors and on vector operations.

Interpretation 1A.9. Direction Similarity and Reversal

Let $\mathbf{v} \in \mathbb{R}^n$ be a nonzero vector. If $k > 0$, then $k\mathbf{v}$ points in the **same direction** as $\mathbf{v}$. If $k < 0$, then $k\mathbf{v}$ points in the **opposite direction** as $\mathbf{v}$.

Interpretation 1A.10. Parallel Vectors.

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors. Then, $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if and only if there exists $k \in \mathbb{R}$ with $k \neq 0$ such that $\mathbf{u} = k\mathbf{v}$. In other words, $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if and only if $\mathbf{u}$ is a nonzero scalar multiple of $\mathbf{v}$.

Observe that from that statement, the zero vector is parallel to every other vector. Therefore, for most applications, we often need to specify that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

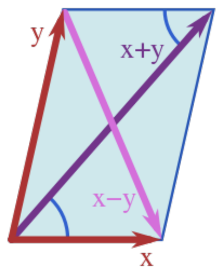
Interpretation 1A.11. The Parallelogram Rule.

Let $\mathbf{u} = (u_1, u_2, ..., u_n), \mathbf{v} = (v_1, v_2, ..., v_n)$ be points in $\mathbb{R}^n$ or equivalently, arrows from the origin to said points. The **Parallelogram Rule** states that the vector $\mathbf{u} + \mathbf{v}$ is the point resulting from traveling from the origin $\mathbf{0}$ in the direction to $\mathbf{u}$ and then in the direction of $\mathbf{v}$.

Interpretation 1A.12. The Triangle Rule.

Let $\mathbf{u} = (u_1, u_2, ..., u_n), \mathbf{v} = (v_1, v_2, ..., v_n)$ be points in $\mathbb{R}^n$ or equivalently, arrows from the origin to said points. The **Triangle Rule** states that the vector $\mathbf{u} - \mathbf{v}$ represents the arrow/direction from $\mathbf{v}$ to $\mathbf{u}$. In other words, let P, Q be points on $\mathbb{R}^n$ with coordinates $\mathbf{p}, \mathbf{q}$ respectively, Then, the vector $\mathbf{q} - \mathbf{p}$ represents the arrow $\overrightarrow{PQ}$.

The image below represents the previous two interpretations on the vectors $\mathbf{x}$ and $\mathbf{y}$:



Part 3. Operations on Vectors in $\mathbb{R}^n$.

We'll talk about four operations on vectors on $\mathbb{R}^n$: (1) norm/magnitude, (2) the dot product, (3) projections, and (4) the cross product. Note that these operations are outside the vector space structure and are inherent to $\mathbb{R}^n$.

To start with, the length of a vector in $\mathbb{R}^n$.

Definition 1A.13. The Norm of a Vector in $\mathbb{R}^n$

Let $\mathbf{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$. Then, the **norm** of $\mathbf{v}$, denoted $||\mathbf{v}||$ or sometimes $|\mathbf{v}|$, is given by

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}.$$

The norm of a vector is also called the **magnitude** or **length** of said vector.

Theorem 1A.14. Properties of Norms in $\mathbb{R}^n$

- **(a) (Absolute Homogeneity)** For any vector $\mathbf{v} \in \mathbb{R}^n$ and scalar $k \in \mathbb{R}$, $||k\mathbf{v}|| = |k|||\mathbf{v}||$.
- **(b) (Positivity)** For any vector $\mathbf{u}$, $||\mathbf{u}|| \geq 0$.
- **(c) (Positive-Definiteness)** $\mathbf{v} = \mathbf{0}$ if and only if $||\mathbf{v}|| = 0$.
- **(d) (The Triangle Inequality)** For any vector $\mathbf{u}, \mathbf{v}$, $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.

With a notion of length, we can define what a unit vector is.

Definition 1A.15. Unit Vectors

A vector $\mathbf{v} \in \mathbb{R}^n$ is a **unit vector** if and only if $||\mathbf{v}|| = 1$.

We give vectors of length 1 a name since they can be very useful in calculations as you'll see later on. For example, the standard basis vectors for $\mathbb{R}^2$ and $\mathbb{R}^3$ are all unit vectors.

Corollary 1A.16. Finding Unit Vectors

Let $\mathbf{v} \in \mathbb{R}^n$ be a nonzero vector. Then, $\dfrac{1}{||\mathbf{v}||}\mathbf{v}$ is a unit vector.

Then, the dot product on $\mathbb{R}^n$.

Definition 1A.17. The Dot Product in $\mathbb{R}^n$

Let $\mathbf{u} = (u_1, u_2, ..., u_n), \mathbf{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$. Then, the **dot product** of $\mathbf{u}$ and $\mathbf{v}$, denoted as $\mathbf{u} \cdot \mathbf{v}$ or as $\langle \mathbf{u}, \mathbf{v} \rangle$, is given by

$$\mathbf{u} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + ... + u_n v_n.$$

Observe that the dot product takes in two vectors as input and returns a **scalar as output**.

While I personally prefer the notation $\langle \mathbf{u}, \mathbf{v} \rangle$, the text uses the dot notation. It may look like the notation $\mathbf{u} \cdot \mathbf{v}$ may be ambiguous when the expression is mixed in with other things, e.g. does $k\mathbf{u} \cdot \mathbf{v}$ mean $(k\mathbf{u}) \cdot \mathbf{v}$ or $k(\mathbf{u} \cdot \mathbf{v})$?, the theorem below allow us to ignore about such ambiguity since the result will be the same regardless. Note that these properties doesn't apply to all vector operations.

Corollary 1A.18. Equality of The Dot Product and the Squared Norm on $\mathbb{R}^n$

$$||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle \text{ for all } \mathbf{v} \in \mathbb{R}^n.$$

Theorem 1A.19. The Properties of the Dot Product in $\mathbb{R}^n$

DP1. Commutativity. For all vectors $\mathbf{u}, \mathbf{v}$: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

DP2. Distributivity over Vector Addition. For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

DP3. Compatibility with Scalar Multiplication. For all vectors $\mathbf{u}, \mathbf{v}$ and scalars k, $\langle k\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$.

DP4. Zero Identity. For all vectors $\mathbf{u}$, $\langle \mathbf{u}, \mathbf{0} \rangle = 0$.

Lastly, we give a geometric identity.

Theorem 1A.20. Dot Product and Angles between Vectors

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Let θ be a determination of the angle between $\mathbf{u}$ and $\mathbf{v}$. Then, $\langle \mathbf{u}, \mathbf{v} \rangle = ||\mathbf{u}||\,||\mathbf{v}|| \cos\theta$.

Observe that we state that θ is **some** determination of the angle between $\mathbf{u}$ and $\mathbf{v}$. That is, it doesn't matter whether the angle is measured counterclockwise or clockwise or whether we start measuring from $\mathbf{u}$ or from $\mathbf{v}$, the identity still applies. To confirm this, recall the properties of the cosine function.

Corollary 1A.21. Parallelism in terms of the Dot Product

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors. Then, $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = \pm||\mathbf{u}||\,||\mathbf{v}||$. More specifically, (1) $\mathbf{u}$ and $\mathbf{v}$ point in the same direction if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = ||\mathbf{u}||\,||\mathbf{v}||$; and (2) $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = (-1)||\mathbf{u}||\,||\mathbf{v}||$.

This is an immediate result from the theorem since for parallel vectors, θ is either $\theta = 0$ or $\theta = \pi$.

The previous theorem motivates the following definition.

Definition 1A.22. Orthogonality

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. We say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, denoted as $\mathbf{u} \perp \mathbf{v}$, if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Additionally, a set of vectors $V = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is orthogonal if and only if V is pairwise orthogonal. That is, for any pairs $\mathbf{v_i}, \mathbf{v_j}$ with $i \neq j$, $\mathbf{v_i} \perp \mathbf{v_j}$.

Observe that the definition of orthogonal does not exclude the zero vector and by properties of the dot product, we can conclude that the zero vector is always orthogonal to any vector. For this reason, we usually make sure to describe vectors being **nonzero** and orthogonal when necessary.

With the dot product above, we can calculate projections of vectors along other vectors. But first, we need to define what a projection is.

Definition 1A.23. (Orthogonal) Projections and Components

Let $\mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{e} \in \mathbb{R}^n$ be a unit vector. The **component** $\text{comp}_{\mathbf{e}}\mathbf{v}$ of $\mathbf{v}$ along $\mathbf{e}$ is the unique scalar $k \in \mathbb{R}$ such that $\mathbf{v} - k\mathbf{e} \perp \mathbf{e}$. The vector $k\mathbf{e}$ is called the **(orthogonal) projection** $\text{proj}_{\mathbf{e}}\mathbf{v}$ of $\mathbf{v}$ along $\mathbf{e}$.

This definition can be extended for a non-unit vector $\mathbf{a} \in \mathbb{R}^n$ by considering the unit vector $\frac{1}{||\mathbf{a}||}\mathbf{a}$, that is,

$$\text{comp}_{\mathbf{a}}\mathbf{v} = k \text{ such that } \left(\mathbf{v} - (k)\frac{1}{||\mathbf{a}||}\mathbf{a}\right) \perp \mathbf{a} \qquad \text{and} \qquad \text{proj}_{\mathbf{a}}\mathbf{v} = (k)\frac{1}{||\mathbf{a}||}\mathbf{a}.$$

Observe that the definition above only works when $\mathbf{a} \neq 0$ and that when $\mathbf{v} = \mathbf{0}$, we can immediately conclude that $k = 0$. The main reason why we consider projections is that it allows us to decompose vectors into orthogonal components. We write orthogonal in parenthesis since we'll consider other projections later when we discuss transformations on $\mathbb{R}^n$. However, when talk of projections **along a vector** (i.e. we only care about **direction**), we always mean the above definition.

Theorem 1A.24. Projections and Components in terms of the Dot Product

Let $\mathbf{v} \in \mathbb{R}^n$ be a vector and $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector. Then, the component and projection of $\mathbf{v}$ along $\mathbf{a}$ are given by the following formulas:

$$\text{comp}_{\mathbf{a}}\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{a} \rangle}{||\mathbf{a}||} \qquad \text{and} \qquad \text{proj}_{\mathbf{a}}\mathbf{v} = (\text{comp}_{\mathbf{a}}\mathbf{v})\frac{\mathbf{a}}{||\mathbf{a}||} = \frac{\langle \mathbf{v}, \mathbf{a} \rangle}{||\mathbf{a}||^2}\mathbf{a} = \frac{\langle \mathbf{v}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}\mathbf{a}$$

Lastly, we introduce the cross product on $\mathbb{R}^3$. Observe that the cross product is only defined on $\mathbb{R}^3$ and not generally on $\mathbb{R}^n$. Therefore, when we invoke the cross product in this course, there is the assumption that we're working in $\mathbb{R}^3$.

Definition 1A.25. Cross Product on $\mathbb{R}^3$

Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ be vectors. The **cross product** $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (-u_1v_3 + u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Do observe that unlike the dot product, **the cross product results in a vector as output**.

Observe that unlike the definitions given above, the definition of the cross product is determined by a formula (instead of the formula being a theorem). While it is possible to define the cross product by its properties, we won't do that in this course. We'll only use the cross product to find normal vectors of planes as listed in a theorem later since this course focuses on matrix algebra. If you were to take a course on vector calculus (e.g. MTH 254 at Oregon State), then you will talk about the cross product in more detail.

Additionally, the formula for the cross product involves something called the **determinant** of a matrix A, denoted by $\det(A)$. We will cover more of that term later in the course after we cover linear transformations. For now, you can ignore the middle part of the equation.

We can give a characterization of parallel vectors using the cross product.

Theorem 1A.26. Parallel Vectors under the Cross Product

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be vectors. Then, $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The previous result gives us that for all vectors $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ since any vector is parallel to itself.

Lastly, we include some properties of the cross product that may help us in calculation.

Theorem 1A.27. Properties of the Cross Product

- **(a) Anti-Commutativity.** $\mathbf{u} \times \mathbf{v} = (-1)(\mathbf{v} \times \mathbf{u})$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.
- **(b) Compatibility with Scalar Multiplication.** $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and scalars $k \in \mathbb{R}$.
- **(c) Distributivity under Vector Addition.** $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$.

There are more properties/results involving the cross product (e.g. the cross product is not associative but you can use something called the Jacobi identity). If you're interested, you can use the corresponding Wikipedia article (linked here) to learn more. You shouldn't need more than the properties I've listed above for this course.

Part 4. Representations of Lines and Planes.

The interpretation of vectors as points lets us describe lines on $\mathbb{R}^2$ and $\mathbb{R}^3$ in terms of vector expressions. Before we start, we want to clarify some terminology.

Convention 1A.28. Parametric Functions and Implicit Equations

In this course, there are generally two ways we can describe a set of points in $\mathbb{R}^n$.

- **(a)** A **parametric function** or a **parametrization** describes a set of points as an image of some function. Here, the image of a function is the set of all outputs based on the range of possible inputs.
- **(b)** An **implicit equation** on $\mathbb{R}^n$ describes a set of points by testing all points on $\mathbb{R}^n$ on some equation or some set of equations. A point is part of the set if and only if the evaluation of the equation on said point results in a true statement. We may also say that the implicit equation is the **equation** of the set of points.

You've seen applications of these before, although you may have not said the terms explicitly.

Example 1A.28.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then, the graph of $f(x)$ is the set of points $(x, f(x))$ on the Cartesian plane. In other words, the **parametric equation** $F : \mathbb{R} \to \mathbb{R}^2$ by $x \mapsto (x, f(x))$

is a parametric function for the graph of $f(x)$.

Example 1A.28.2. Let $r > 0$. The graph of the function $f(x) = \sqrt{r^2 - x^2}$ is the upper-half circle on the Cartesian plane centered at the origin with radius r.

Example 1A.28.3. Let $r > 0$. The equation $(x-h)^2 + (y-k)^2 = r^2$ is an **implicit equation** that describes a circle on $\mathbb{R}^2$ centered at the point (h, k) with radius r. Here, we refer to coordinates in $\mathbb{R}^2$ as (x, y).

The following results will be about lines.

Theorem 1A.29. Parametrization of Lines

Let L be a line on $\mathbb{R}^2$. Let $\mathbf{r_0}$ represent any point on the line and let $\mathbf{d}$ represent an arrow parallel to the line. Often, we call the vector $\mathbf{d}$ the **direction vector** of the line. Then, the parametric equation $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ given by

$$\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{d}$$

describes the line L. Furthermore, the vector $\mathbf{d}$ is unique up to multiplication by a nonzero scalar. This result applies to $\mathbb{R}^3$ as well.

Example 1A.29.1. The x-axis on $\mathbb{R}^2$ can be described using the parametric function $\mathbf{r}(t) = t(1, 0)$. Similarly, the y-axis admits a parametrization of $\mathbf{s}(t) = t(0, 1)$.

Corollary 1A.30. Parametrization of Lines by $y = mx + b$

Let L be a line on $\mathbb{R}^2$. The line L is the graph of the function $y = mx + b$ with m the slope of L and b the y-intercept if and only if L admits the following parametrization:

$$\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{d} \text{ with } \mathbf{r_0} = (0, b), \mathbf{d} = (1, m).$$

This result also applies for $\mathbf{d} = (x_0, y_0)$ if the slope m can be expressed as $m = \dfrac{y_0}{x_0}$ since the direction vector $\mathbf{d}$ and the slope m are both unique up to multiplication by a nonzero scalar. In other words, the slope of L determines its direction vector and vice-versa.

Corollary 1A.31. Two points define a line

Let L be a line and let $\mathbf{x}, \mathbf{y}$ represent distinct points in L. Then, L admits the following parametrization:

$$\mathbf{r}(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x}).$$

In other words, the vector $\mathbf{y} - \mathbf{x}$ results in the direction vector of L.

This result is similar to how two only need two points to express a line in point-slope form.

The following results will be about planes.

Theorem 1A.32. Parametrization of Planes

Let P be a plane on $\mathbb{R}^3$. Let $\mathbf{p_0}$ represent any point on the plane and let $\mathbf{d_1}, \mathbf{d_2}$ represent two vectors parallel to the plane but not parallel to each other. Then, P admits the parametrization $\mathbf{p} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\mathbf{p}(t, s) = \mathbf{p_0} + t\mathbf{d_1} + s\mathbf{d_2}.$$

We can call the vectors $\mathbf{d_1}$ and $\mathbf{d_2}$ direction vectors of the plane. These direction vectors are not uniquely identified by the plane.

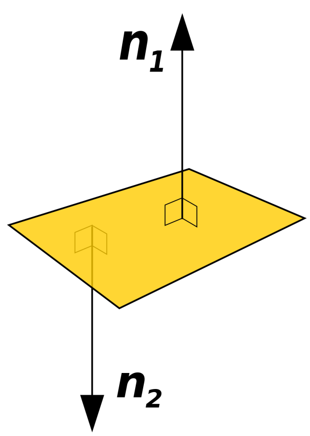
Example 1A.32.1. The xy-axis on $\mathbb{R}^3$ admits the parametrization $\mathbf{p}(t, s) = (0, 0, 0) + t(1, 0, 0) + s(0, 1, 0)$. It also admits the parametrization $\mathbf{q}(t, s) = (0, 0, 0) + t(1, 1, 0) + s(-1, 1, 0)$.

For this course, we often will not make use of this parametrization of the plane because there are an infinite number of choices for the vectors $\mathbf{d_1}$ and $\mathbf{d_2}$ (even considering nonzero multiples) as shown in the example above. However, there is a representation of the plane that is, in some sense, unique.

Theorem 1A.33. Normal Vectors

Let P be a plane on $\mathbb{R}^3$. Then, there exists a nonzero vector $\mathbf{N}$ such that the vector $\mathbf{N}$ is perpendicular to $\mathbf{p} - \mathbf{q}$ for all vectors $\mathbf{p}, \mathbf{q} \in P$, i.e. the vector $\mathbf{N}$ is perpendicular to any direction vector of P. Furthermore, the vector $\mathbf{N}$ is unique up to multiplication by a nonzero scalar. We call the vector $\mathbf{N}$ to be the **normal vector** of P.

Please see below for an illustration. Observe that the normal vectors presented below labeled $\mathbf{n_1}$ and $\mathbf{n_2}$ are scalar multiples of each other. In this case, $\mathbf{n_2} = k\mathbf{n_1}$ for some $k \in \mathbb{R}$ negative. Also, observe that normal vectors also represent direction, i.e. they don't carry position information of the plane. A starting point is required. This is similar to how the direction vector $\mathbf{d}$ of a line only carries direction information and you need an initial position $\mathbf{r_0}$ to determine the line.



The two results below are tools we can use the find this representation.

Corollary 1A.34. Calculation of Normal Vectors

Given two non-parallel direction vectors $\mathbf{d_1}, \mathbf{d_2}$ of a plane P, the normal vector $\mathbf{N}$ of P is given by $\mathbf{N} = \mathbf{d_1} \times \mathbf{d_2}$ where $\times$ is the cross product.

Corollary 1A.35. Equations of Planes

Let P be a plane in $\mathbb{R}^3$. Let $\mathbf{p_0} = (x_0, y_0, z_0)$ be a point in P and let $N = (N_x, N_y, N_z)$ be the normal vector of P. Denote points in $\mathbb{R}^3$ as $\mathbf{x} = (x, y, z)$. Then, P is described by the implicit equation

$$\mathbf{N} \cdot (\mathbf{x} - \mathbf{p_0}) = 0 \qquad \text{or equivalently} \qquad N_x(x - x_0) + N_y(y - y_0) + N_z(z - z_0) = 0.$$

Observe that $\mathbf{x} - \mathbf{p_0}$ describes a direction vector of P and since $\mathbf{x}$ can stand for any vector on the plane, it represents all possible direction vectors.